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# CONSERVATION EQUATIONS FOR A NONSTEADY FLOW OF A COMPRESSIBLE VISCOUS SINGLE-PHASE FLUID IN VARIOUS COORDINATE SYSTEMS

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Based on a generalization of the Reynolds transport theorem the mass, momentum and energy conservation equations for a nonsteady flow of a compressible viscous single-phase fluid (expressed in vector and dyadic notations) are first formulated in integral form (in terms of moving volume and surface elements), next converted into differential form and then transformed in terms of orthogonal curvilinear coordinates. Specialized equations are obtained for three-dimensional flows in Cartesian, cylindrical and spherical coordinates. These equations can then be reduced to corresponding equations for one- and two-dimensional flows in various coordinates. Equations for the vorticity, entropy and enthalpy and Bernoulli equation are also summarized.

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18. SUBJECT TERMS (Continued)

Energy Conservation Equation
Cartesian, Cylindrical, Spherical, and Orthogonal Curvilinear Coordinates
One, Two, and Three Dimensional Flows
Vector and Dyadic (Second Order Tensor) Notations
Bernoulli Equation
Vorticity
Entropy
Enthalpy
Kinematic Transport Theorem
Integral and Differential Formulations
Conservative and Nonconservative Forms

TABLE OF CONTENTS

# SECTION 1

## INTRODUCTION

The governing equations in fluid mechanics are usually expressed in terms of Cartesian coordinates. They are not readily convertible into forms useful for practical applications to one-, two- and three-dimensional cylindrical and spherical problems. One of the purposes of this report is to transform the mass, momentum and energy conservation equations for a nonsteady flow of a compressible viscous single-phase fluid from expressions in vector and dyadic notations to those in terms of orthogonal cuvilinear coordinates and then to specialize the equations for Cartesian, cylindrical and spherical coordinates.

Another purpose of this report is to provide a theoretical basis for usage of Eulerian sliding grids in modern computational fluid mechanics by establishing a kinematic transport theorem which is a generalization of the Reynolds transport theorem. Based upon the theorem the conservation equations in integral form are first formulated in terms of moving volume and surface elements and then converted into the corresponding equations in differential form by using the generalized definition of the vector operator $\nabla$.

In order to present an overall survey of the basic equations this report also summarizes the equations for the vorticity, entropy and enthalpy and Bernoulli equation.

## SECTION 2

## INTEGRAL FORMULATION

### 2.1 KINEMATIC TRANSPORT THEOREM.

Let $\underline{x} = (x_1, x_2, x_3)$ denote the rectangular spatial ("Eulerian") coordinates which identify a fixed point in space. Let $\underline{X} = (X_1, X_2, X_3)$ denote the rectangular material ("Lagrangian") coordinates which identify a fluid particle in motion. Let $F = F(\underline{x}, t)$ represent any arbitrary single-valued scalar or vector point function (of position $\underline{x}$ and time t) possessing continuous derivatives. The function

$$M_{\overline{V}} = \int_{\overline{V}} F(\underline{x}, t)d\overline{V} = \int_{V_o} F[\underline{x}(\underline{X}, t), t]J\, dV_o , \qquad (1)$$

where $\overline{V} = \overline{V}(t)$ denotes a material volume (that is, a volume moving with the fluid), is a well-defined function of time. In Eq. (1) the Jacobian (Ref. 1, p.33)

$$J = \frac{d\overline{V}}{dV_o} = \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} = \det\left(\frac{\partial x_i}{\partial X_\alpha}\right), \quad (i = 1,2,3;\ \alpha = 1,2,3) \qquad (2)$$

relates the element $d\overline{V}$ of the moving volume $\overline{V}$ in the x-variables to the element $dV_o$ of the fixed volume $V_o = \overline{V}(0) = \overline{V}(t)$ as $t = 0$ in the X-variables.

Using Euler's expansion formula (Ref. 2)

$$\frac{dJ}{dt} = J\nabla \cdot \underline{v} \qquad (3)$$

(where $\underline{v}$ is the velocity vector of the fluid motion) and the relation between the material time derivative and the spatial derivatives (Ref. 3, Eq. (3.6))

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \nabla \qquad (4)$$

we can express the material time derivative of $M_{\bar{V}}$ as

$$\begin{aligned} \frac{d}{dt} M_{\bar{V}} &= \int_{V_o} \left( J \frac{dF}{dt} + F \frac{dJ}{dt} \right) dV_o \\ &= \int_{V_o} \left[ \frac{dF}{dt} + F(\nabla \cdot \underline{v}) \right] J dV_o \\ &= \int_{\bar{V}} \left[ \frac{\partial F}{\partial t} + \underline{v} \cdot \nabla F + F(\nabla \cdot \underline{v}) \right] d\bar{V} \\ &= \int_{\bar{V}} \left[ \frac{\partial F}{\partial t} + \mathrm{div}\ (F\ \underline{v}) \right] d\bar{V} \quad . \end{aligned} \qquad (5)$$

Then in view of Green's transformation (Ref. 1, Eq. (7.2)) for any vector or tensor field $\phi$

$$\int_V \mathrm{div}\ \phi\ dV = \oint_A d\underline{A} \cdot \phi \ , \ (d\underline{A} = \underline{n} dA) \qquad (6)$$

we obtain

$$\int_V \frac{\partial F}{\partial t} dV = -\oint_A F\underline{v}\cdot\underline{n}dA + \frac{d}{dt}M_{\bar{V}} \quad , \tag{7}$$

or

$$\frac{\partial}{\partial t}\int_V F dV = -\oint_A F\underline{v}\cdot\underline{n}dA + \frac{d}{dt}M_{\bar{V}} \quad , \tag{7a}$$

where V denotes the volume fixed in space which instantaneously coincides with the material volume $\bar{V}$, A denotes the surface bounding the volume V and $\underline{n}$ denotes the unit vector along the outward normal to A. Eq. (7a) is known as the transport theorem of Reynolds (Ref. 1, Eq. (25.4) and Ref. 4, § 14). It is a kinematic relation independent of any meaning attached to F. All fluid physics is contained in the $\frac{d}{dt}M_{\bar{V}}$ term.

Now let us consider a volume $\tilde{V} = \tilde{V}(t)$ with bounding surface $\tilde{A} = \tilde{A}(t)$ sliding with respect to the fluid and obeying the relation

$$\frac{\partial}{\partial t}(d\tilde{V}) = \underline{u}\cdot\underline{n}d\tilde{A} \quad , \tag{8}$$

where $\underline{u}$ is the local surface velocity. We can write

$$\int_{\tilde{V}} F\frac{\partial}{\partial t}(d\tilde{V}) = \oint_{\tilde{A}} F\underline{u}\cdot\underline{n}d\tilde{A} \quad . \tag{9}$$

To each time instant t when the fixed volume V coincides with the material volume $\bar{V}$ there corresponds a time instant $t_1$ when the moving volume $\tilde{V}$ coincides with the same material volume $\bar{V}$ instantaneously. At $t_1$ Eq. (9) becomes

$$\int_V F_1 \frac{\partial}{\partial t_1}(d\tilde{V}) = \oint_A F_1\underline{u}_1\cdot\underline{n}d\tilde{A} \quad , \tag{10}$$

where $F_1 = F(\underline{x},t_1)$, $\underline{u}_1 = \underline{u}(\underline{x},t_1)$ and $\frac{\partial}{\partial t_1}(d\tilde{V}) = \left[\frac{\partial}{\partial t}(d\tilde{V})\right]_{t = t_1}$.

Since V, A and $\underline{n}$ in Eq. (10) have the same meaning as those in Eq. (7) and dV and dA (at t) in Eq. (7) equal to $d\tilde{V}$ and $d\tilde{A}$ (at $t_1$) in Eq. (10) we may add Eqs. (7) and (10) to obtain

$$\int_V \left[\frac{\partial F}{\partial t} d\tilde{V} + F_1 \frac{\partial}{\partial t_1}(d\tilde{V})\right] = - \oint_A (F\underline{v} - F_1\underline{u}_1)\cdot\underline{n}d\tilde{A} + \frac{d}{dt} M_{\overline{V}} \ . \qquad (11)$$

In case V and $\tilde{V}$ can coincide with $\overline{V}$ at the same time ($t_1 = t$) (for example, if $\underline{u}_1 = \underline{0}$ or $\underline{u}_1 = \underline{v}$) Eq. (11) becomes

$$\int_V \frac{\partial}{\partial t}(Fd\tilde{V}) = - \oint_A F(\underline{v} - \underline{u})\cdot\underline{n}d\tilde{A} + \frac{d}{dt}M_{\overline{V}} \quad , \qquad (11a)$$

or (since V is fixed in space)

$$\frac{\partial}{\partial t}\int_V Fd\tilde{V} = - \oint_A F(\underline{v} - \underline{u})\cdot\underline{n}d\tilde{A} + \frac{d}{dt}M_{\overline{V}} \quad . \qquad (11b)$$

This is a generalization of the Reynolds transport theorem and gives the fundamental transport formula for the time variation of the volume integral of any fluid state quantity F over a moving volume which instantaneously coincides with the material volume. As $\underline{u} = \underline{0}$ ($\tilde{V} = V$, hence $t_1 = t$) Eq. (11b) reduces to Eq. (7a) which is the basis of the Eulerian grid representation in numerical algorithm. As $\underline{u} = \underline{v}$ ($\tilde{V} = \overline{V}$, hence $t_1 = t$) Eq. (11b) becomes a statement of the Lagrangian mesh representation. Ref. 5 shows an example of the application of Eq. (11b).

## 2.2 CONSERVATION LAWS.

The mass, momentum and energy conservation equations for a non-steady flow of a compressible, viscous, heat-conducting fluid

derived and discussed in reference books on gas dynamics (see, for example, Refs. 6 to 13) have been summarized in recent books on computational fluid mechanics (see, for example, Refs. 14 to 16) in compact vector and dyadic (second order tensor) notations. In a fixed Eulerian frame of reference these equations in integral form (which is more basic than the differential form in expressing the physical laws) are as follows:

$$\underline{\text{mass}}: \quad \frac{\partial}{\partial t}\int_V \rho dV + \oint_A \rho\underline{v}\cdot\underline{n}dA = 0 \quad , \tag{12}$$

$$\underline{\text{momentum}}: \quad \frac{\partial}{\partial t}\int_V \rho\underline{v}dV + \oint_A \rho\underline{v}(\underline{v}\cdot\underline{n})dA = \oint_A \underline{\underline{\sigma}}\cdot\underline{n}dA + \int_V \underline{f}dV \quad , \tag{13}$$

$$\underline{\text{energy}}: \quad \frac{\partial}{\partial t}\int_V \rho EdV + \oint_A \rho E\underline{v}\cdot\underline{n}dA = \oint_A (\underline{\underline{\sigma}}\cdot\underline{v} - \underline{q})\cdot\underline{n}dA + \int_V \underline{f}\cdot\underline{v}dV + \int_V GdV \quad . \tag{14}$$

In these equations, $\rho$ is the density, E the total specific energy ("specific" means per unit mass)

$$E = e + \frac{1}{2} v^2 \tag{15}$$

(where e is the specific internal energy), $\underline{\underline{\sigma}}$ is the stress tensor, $\underline{q}$ the heat-flux vector, $\underline{f}$ the external force vector per unit volume, and G is the energy generation per unit volume, and V is a control volume fixed in space, A is the surface bounding V, and $\underline{n}$ is the unit vector along the outward normal to A.

In these equations the properties of the fluid need not be continuous functions of space and time.

By comparison of Eq. (7a) with Eqs. (12) to (14) the function F may be identified as $\rho$, $\rho \underline{v}$ and $\rho E$ respectively and the term $dM_{\overline{V}}/dt$ as the right-hand side terms of each of these equations. Then in terms of the moving elements ($d\tilde{V} = dV$ and $d\tilde{A} = dA$ instantaneously) the mass, momentum and energy conservation equations in case $t_1 = t$ (exactly or approximately) can be expressed exactly or approximately as a single vector equation according to Eq. (11b):

$$\frac{\partial}{\partial t} \int_V \underline{W}\, d\tilde{V} = - \oint_A \underline{W}(\underline{v} - \underline{u}) \cdot \underline{n} d\tilde{A} + \oint_A \underline{\Sigma} \cdot \underline{n} d\tilde{A} + \int_V \underline{Q} d\tilde{V} \quad , \tag{16}$$

where

$$\underline{W} = \begin{pmatrix} \rho \\ \rho \underline{v} \\ \rho E \end{pmatrix}, \quad \underline{\Sigma} = \begin{pmatrix} 0 \\ \underline{\underline{\sigma}} \\ \underline{\underline{\sigma}} \cdot \underline{v} - \underline{q} \end{pmatrix}, \quad \underline{Q} = \begin{pmatrix} 0 \\ \underline{f} \\ \underline{f} \cdot \underline{v} + G \end{pmatrix} \quad . \tag{17}$$

Eq. (16) provides a theoretical basis for usage of Eulerian sliding grids in finite-difference numerical algorithm (see, for example, Ref. 5).

## 2.3 CONSTITUTIVE RELATIONS.

The basic dependent variables in Eqs. (12) to (14) or Eqs. (16) are $\rho$, $\underline{v}$ and E (or e). Constitutive relations for $\underline{\underline{\sigma}}$ and $\underline{q}$ must be added to these equations in order to obtain a closed system. We are concerned here with the case of Newtonian fluids, i.e., by definition, fluids such that the stress tensor

is a linear function of the velocity gradient. From this definition, excluding the existence of distributed force couples, results Newton's law, also called the Navier-Stokes law, for $\underline{\underline{\sigma}}$:

$$\underline{\underline{\sigma}} = - p\underline{\underline{I}} + \underline{\underline{\tau}} \quad , \tag{18}$$

with

$$\underline{\underline{\tau}} = \lambda \ (\nabla\cdot\underline{v})\underline{\underline{I}} + 2\mu\underline{\underline{D}} \tag{19}$$

and

$$\underline{\underline{D}} = \frac{1}{2}\left[\nabla\underline{v} + (\nabla\underline{v})^{t}\right] \quad , \tag{20}$$

the superscript t denoting the transpose of a tensor. In Eqs. (18) to (20) p is the pressure, $\underline{\underline{\tau}}$ is the viscous (or deviatoric) stress tensor, $\underline{\underline{I}}$ is the unit tensor, $\mu$ and $\lambda$ are the first (shear) and second (dilatational) coefficients of viscosity, and $\underline{\underline{D}}$ is the tensor of rates of deformation. Furthermore, the fluid is assumed to obey Fourier's law of heat conduction for $\underline{q}$:

$$\underline{q} = -k\nabla T \quad , \tag{21}$$

where T is the absolute temperature, and k is the thermal conductivity coefficient. Many fluids, in particular air and water, follow Newton's law and Fourier's law.

## 2.4 THERMODYNAMIC/TRANSPORT PROPERTIES.

The state variables $\rho$, e, p, T and the specific entropy S are connected by thermodynamic relations (assuming local thermodynamic equilibrium).

We consider the case of a simple fluid such that all its thermodynamic properties can be deduced from a single fundamental relationship which, for a compressible fluid, can be chosen of the type

$$S = S(\rho,e) \quad . \tag{22}$$

From this relationship p and T are obtained in terms of the basic variables $\rho$ and e from

$$p = -\rho^2 T\left(\frac{\partial S}{\partial \rho}\right)_e \ , \ T = \frac{1}{(\partial S/\partial e)_\rho} \quad . \tag{23}$$

An important special case is a perfect gas with constant specific heats $c_p$ and $c_v$. For such a gas the laws of state are

$$p = (\gamma-1)\rho e, \ (\gamma = c_p/c_v) \tag{24}$$

and

$$e = c_v T \quad . \tag{25}$$

The viscosity and thermal conductivity coefficients depend on the local thermodynamic state; in most conditions they depend only on the temperature:

$$\mu = \mu(T), \quad \lambda = \lambda(T), \ k = k(T) \quad . \tag{26}$$

The coefficient

$$\kappa = 3\lambda + 2\mu \tag{27}$$

is called bulk viscosity coefficient In the "Stokes relation" (Ref. 3, p. 238)

$$3\lambda + 2\mu = 0 \tag{28}$$

it is assumed to be zero. However, except for very special conditions, for example, monatomic gases, there is no reason to assume $3\lambda = -2\mu$. (Ref. 9, p.337 and Ref. 7, p.540.)

## SECTION 3

## CONSERVATION LAWS IN DIFFERENTIAL FORM

### 3.1 CONVERSION BY GENERALIZED DEFINITION OF OPERATOR $\nabla$.

If the properties of the fluid are continuous and sufficiently differentiable in some domain of space and time, then the conservation equations in integral form can be converted into an equivalent set of partial differential equations through a generalized definition of the vector operator $\nabla$ (Ref. 17, p.40)

$$\nabla\phi = \lim_{V\to 0} \frac{1}{V} \oint_A \phi \underline{n} dA \ , \tag{29}$$

where $\phi$ is an unspecified (scalar, vector or dyadic) function of position, V is the volume enclosed by a surface A to which the point P at which $\nabla\phi$ is to be calculated remains interior, while the largest dimension of A tends to zero, and the multiplication in $\nabla\phi$ may be scalar, vector or dyadic. In particular when $\phi = \underline{b}$, a vector, the scalar product becomes

$$\nabla\cdot\underline{b} = \lim_{V\to 0} \frac{1}{V} \oint_A \underline{b}\cdot\underline{n} dA \ , \tag{30}$$

which is a scalar, and when $\phi = \underline{\underline{\beta}}$, a dyadic, the dot product becomes

$$\nabla\cdot\underline{\underline{\beta}} = \lim_{V\to 0} \frac{1}{V} \oint_A \underline{\underline{\beta}}\cdot\underline{n} dA \ , \tag{31}$$

which is a vector. Eqs. (30) and (31) can be also obtained by dividing Eq. (6) by V and then taking the limits of both sides as V approaches to zero.

### 3.2 CONSERVATIVE FORM.

Dividing Eqs. (12) to (14) by V, taking the limit of every term as V approaches zero and applying Eqs. (30) and (31) we obtain the conservation equations in differential form:

<u>mass</u>: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0 \quad , \tag{32}$$

<u>momentum</u>: $$\frac{\partial}{\partial t} (\rho \underline{v}) + \nabla \cdot (\rho \underline{v}\underline{v} - \underline{\underline{g}}) = \underline{f} \quad , \tag{33}$$

<u>energy</u>: $$\frac{\partial}{\partial t} (\rho E) + \nabla \cdot (\rho E \underline{v} - \underline{\underline{g}} \cdot \underline{v} + \underline{q}) = \underline{f} \cdot \underline{v} + G \ . \tag{34}$$

An alternate form of the energy equation Eq. (34) is in terms of the enthalpy per unit mass

$$h = e + p/\rho \tag{35}$$

Substituting

$$E = h + \frac{1}{2} v^2 - p/\rho = H - p/\rho \tag{36}$$

into Eq. (34) we obtain

$$\frac{\partial}{\partial t} (\rho H) + \nabla \cdot (\rho H \underline{v} - p\underline{v} - \underline{\underline{g}} \cdot \underline{v} + \underline{q}) = \underline{f} \cdot \underline{v} + \frac{\partial p}{\partial t} + G \ , \tag{37}$$

where

$$H = h + \frac{1}{2} v^2 = E + p/\rho \tag{38}$$

is the total enthalpy per unit mass (which is also known as the "specific stagnation enthalpy" or for nonviscous and nonheat-conducting fluid in steady motion the "total energy per unit mass", Ref. 18, p. 33).

Substituting Eq. (36) into Eq. (14) we obtain the integral form of the conservation equation for $\rho H$ in terms of fixed elements dV and dA

$$\frac{\partial}{\partial t}\int_V \rho H dV + \oint_A \rho H \underline{v}\cdot\underline{n} dA$$

$$= \oint_A (\underline{\underline{\sigma}}\cdot\underline{v} - \underline{q} + p\underline{v})\cdot\underline{n} dA + \int_V (\underline{f}\cdot\underline{v} + G + \frac{\partial p}{\partial t}) dV \quad . \tag{37a}$$

The corresponding conservation equation in terms of moving elements $d\tilde{V}$ and $d\tilde{A}$ in case $t_1 = t$ is Eq. (16) with

$$\underline{W} = (\rho H)\,, \quad \underline{\Sigma} = (\underline{\underline{\sigma}}\cdot\underline{v} - \underline{q} + p\underline{v})\,, \quad \underline{Q} = (\underline{f}\cdot\underline{v} + G + \frac{\partial p}{\partial t}) \tag{17a}$$

### 3.3 NONCONSERVATIVE FORM.

Eqs. (32) to (34) are mass, momentum and energy equations in "conservative" or "divergence" form. (See Section III-A-3 of Ref. 14 for the meaning and beneficial effects of using this form). The corresponding equations in nonconservative form in which $\underline{\underline{\sigma}}$ follows Newton's law are (Ref. 16, Section 1.1):

mass:
$$\frac{d\rho}{dt} + \rho\nabla\cdot\underline{v} = 0 \quad , \tag{39}$$

momentum:
$$\rho\,\frac{d\underline{v}}{dt} + \nabla p = \underline{f} + \mu\nabla^2\underline{v} + (\lambda+\mu)\nabla(\nabla\cdot\underline{v})$$
$$+ (\nabla\cdot\underline{v})\nabla\lambda + 2\underline{\underline{D}}\cdot\nabla\mu \quad , \tag{40}$$

energy:
$$\rho\,\frac{de}{dt} + p\nabla\cdot\underline{v} = \Phi - \nabla\cdot\underline{q} + G \quad , \tag{41}$$

where $\Phi$ is the dissipation function

$$\Phi = \underline{\underline{\tau}}:\nabla\underline{v} = \lambda(\nabla\cdot\underline{v})^2 + 2\mu\underline{\underline{D}}:\underline{\underline{D}} \tag{42}$$

and $\frac{d}{dt}$ is the material derivative given in Eq. (4).

An alternate form of the energy equation Eq. (41) is in terms of the specific entropy S such that (Ref. 3, Eq. (33.1))

$$TdS = de + pd\left(\frac{1}{\rho}\right) \quad . \tag{43}$$

By use of Eqs. (39) and (43) we obtain from Eq. (41)

$$\rho T \frac{dS}{dt} = \Phi - \nabla\cdot\underline{q} + G \quad . \tag{44}$$

Adding Eq. (32) (multiplied by S) and Eq. (44) (divided by T) we have the conservative form of the conservation equation for $\rho S$

$$\frac{\partial}{\partial t}(\rho S) + \nabla\cdot(\rho S\underline{v}) = \frac{1}{T}(\Phi - \nabla\cdot\underline{q} + G) \quad . \tag{44a}$$

Taking the integral of Eq. (44a) over a fixed control volume V and applying Green's transformation Eq. (6) we obtain the integral form of the conservation equation for $\rho S$ in terms of fixed elements dV and dA

$$\frac{\partial}{\partial t}\int_V \rho S dV + \oint_A \rho S\underline{v}\cdot\underline{n}dA$$

$$= \int_V \frac{1}{T}(\Phi - \nabla\cdot\underline{q} + G)dV \tag{44b}$$

The corresponding conservation equation in terms of moving elements $d\tilde{V}$ and $d\tilde{A}$ in case $t_1 = t$ is Eq. (16) with

$$\underline{W} = (\rho S),\ \underline{\Sigma} = (\underline{0}),\ \underline{Q} = (\Phi/T - \nabla\cdot\underline{q}/T + G/T) \tag{17b}$$

3.4 VORTICITY DISTRIBUTION.

The vorticity vector

$$\underline{\Omega} = \nabla \times \underline{v} \tag{45}$$

gives the intrinsic rotation of each fluid element.

By taking the curl of both sides of Eq. (40) (divided by $\rho$) and noting that

$$\nabla \cdot \underline{\Omega} = 0 \tag{46}$$

and

$$\nabla \times \nabla \text{ (scalar function)} = \underline{0} \tag{47}$$

we obtain the general equation of vorticity distribution

$$\begin{aligned}\frac{d\underline{\Omega}}{dt} &- (\underline{\Omega}\cdot\nabla)\underline{v} + \underline{\Omega}(\nabla\cdot\underline{v}) + \nabla(\tfrac{1}{\rho}) \times \nabla p \\ &= \frac{1}{\rho}\nabla \times \underline{f} + \nabla(\tfrac{1}{\rho}) \times \underline{f} + \frac{\mu}{\rho}\nabla^2\underline{\Omega} - \nabla(\tfrac{\mu}{\rho}) \times \nabla \times \underline{\Omega} \\ &\quad + \nabla(\tfrac{\lambda+2\mu}{\rho}) \times \nabla(\nabla\cdot\underline{v}) - \frac{1}{\rho}\nabla\lambda \times \nabla(\nabla\cdot\underline{v}) \\ &\quad + (\nabla\cdot\underline{v})\nabla(\tfrac{1}{\rho}) \times \nabla\lambda + \frac{2}{\rho}\nabla \times (\underline{\underline{D}}\cdot\nabla\mu) + \nabla(\tfrac{2}{\rho}) \times (\underline{\underline{D}}\cdot\nabla\mu)\end{aligned} \tag{48}$$

If $\mu$ and $\lambda$ are not functions of space variables Eq. (48) reduces to

$$\frac{d\underline{\Omega}}{dt} - (\underline{\Omega}\cdot\nabla)\underline{v} + \underline{\Omega}(\nabla\cdot\underline{v}) + \nabla\left(\frac{1}{\rho}\right)\times \nabla p$$

$$= \frac{1}{\rho}\nabla \times \underline{f} + \nabla\left(\frac{1}{\rho}\right)\times \underline{f} + \frac{\mu}{\rho}\nabla^2\underline{\Omega}$$

$$+ (\lambda+2\mu)\nabla\left(\frac{1}{\rho}\right) \times \nabla(\nabla\cdot\underline{v}) - \mu\nabla\left(\frac{1}{\rho}\right) \times\nabla\times\underline{\Omega} \tag{49}$$

If the flow is incompressible which is characterized by the condition

$$\nabla\cdot\underline{v} = 0 \tag{50}$$

and implies that $\rho$ is constant along a fluid particle trajectory but not necessarily independent of space variables (as in stratified flows, Ref. 16, Section 1.3) Eq. (48) reduces to

$$\frac{d\underline{\Omega}}{dt} - (\underline{\Omega}\cdot\nabla)\underline{v} + \nabla\left(\frac{1}{\rho}\right) \times \nabla p$$

$$= \frac{1}{\rho}\nabla \times \underline{f} + \nabla\left(\frac{1}{\rho}\right) \times \underline{f} + \frac{\mu}{\rho}\nabla^2\underline{\Omega}$$

$$-\nabla\left(\frac{\mu}{\rho}\right)\times\nabla\times\underline{\Omega} + \frac{2}{\rho}\nabla \times (\underline{\underline{D}}\cdot\nabla\mu) + \nabla\left(\frac{2}{\rho}\right) \times (\underline{\underline{D}}\cdot\nabla\mu)\,. \tag{51}$$

If, in addition, $\rho$ is constant everywhere Eq. (51) reduces to

$$\frac{d\underline{\Omega}}{dt} - (\underline{\Omega}\cdot\nabla)\underline{v}$$

$$= \frac{1}{\rho}\nabla \times \underline{f} + \frac{\mu}{\rho}\nabla^2\underline{\Omega}$$

$$- \frac{1}{\rho}\nabla\mu \times \nabla\times\underline{\Omega} + \frac{2}{\rho}\nabla \times (\underline{\underline{D}}\cdot\nabla\mu)\,. \tag{52}$$

If, in addition, $\mu$ is not a function of space variables Eq. (52) reduces to (Ref. 16, Eq. (1.37))

$$\frac{d\underline{\Omega}}{dt} - (\underline{\Omega}\cdot\nabla)\underline{v} = \frac{1}{\rho}\nabla\times\underline{f} + \frac{\mu}{\rho}\nabla^2\underline{\Omega} \quad . \tag{53}$$

This equation is usually associated with an equation for a solenoid stream-function vector $\underline{\Psi}$ such that

$$\underline{v} = \nabla \times \underline{\Psi} \quad , \tag{54}$$

which automatically satisfies the incompressibility condition Eq. (50). The equation satisfied by $\underline{\Psi}$ is derived by applying the curl operator to Eq. (54) and using definition Eq. (45) to obtain

$$\nabla^2\underline{\Psi} + \underline{\Omega} = \underline{0} \tag{55}$$

since

$$\nabla\cdot\underline{\Psi} = 0 \tag{56}$$

as $\underline{\Psi}$ is solenoid.

Let us return now to consider the case of compressible flow. For an ideal fluid ($\mu=\lambda=0$) either Eq. (48) or Eq. (49) reduces to [Ref. 18, Eq. (7-20)]

$$\frac{d\underline{\Omega}}{dt} - (\underline{\Omega}\cdot\nabla)\underline{v} + \underline{\Omega}(\nabla\cdot\underline{v})$$

$$= \frac{1}{\rho}\nabla\times\underline{f} + \nabla\left(\frac{1}{\rho}\right)\times\underline{f} - \nabla\times\left(\frac{1}{\rho}\nabla p\right) \quad . \tag{57}$$

Another form of this equation can be obtained by using Eq. (43) [Ref. 18, Eq. (7-21)] :

$$\frac{d\underline{\Omega}}{dt} - (\underline{\Omega}\cdot\nabla)\underline{v} + \underline{\Omega}(\nabla\cdot\underline{v})$$

$$= \frac{1}{\rho}\nabla \times \underline{f} + \nabla\left(\frac{1}{\rho}\right) \times \underline{f} + \nabla T \times \nabla S \quad . \tag{58}$$

If p is a function of $\rho$ only the last term of Eq. (57) can be written as

$$\nabla \times \left(\frac{1}{\rho}\nabla p\right) = \nabla \times \nabla\left(\int\frac{dp}{\rho}\right) , \tag{59}$$

which is zero according to Eq. (47). If the specific entropy S is a constant, then the last term of Eq. (58) is zero because $\nabla S = \underline{0}$. Both conditions are satisfied if the fluid is isentropic, i.e., the fluid has the same entropy everywhere. In the absence of any external force $\underline{f}$, if $\underline{\Omega}$ is zero for one instant at every point of the flow field then $d\underline{\Omega}/dt = \underline{0}$. Therefore the vorticity at every point of the field will remain zero and the flow is irrotational. On the other hand, if the flow is not isentropic, i.e., the fluid has different entropy at different points of the field, the last term of Eq. (58) will cause the vorticity to be different from zero in the next instant. Therefore, nonisentropy flows cannot be irrotational. Hence irrotationality implies isentropy, but isentropy does not imply irrotationality.

Multiplying Eq. (48) by $\rho$, expanding the term $(\rho\underline{\Omega}\cdot\nabla)\underline{v}$ by formula (X) on p. 44 of Ref. 17 and adding Eq. (32) (multiplied by $\underline{\Omega}$) we have the conservative form of the conservation equation for $\rho\underline{\Omega}$

$$\frac{\partial}{\partial t}(\rho\underline{\Omega}) + \nabla\cdot(\rho\underline{v}\ \underline{\Omega}) = -\nabla\cdot(\rho\underline{\Omega}\ \underline{v}) + \underline{Q}_1 \tag{48a}$$

where

$$\begin{aligned}\underline{Q}_1 = & -\underline{v}[\nabla\cdot(\rho\underline{\Omega})] - \rho\underline{\Omega}(\nabla\cdot\underline{v}) - \rho\nabla(\frac{1}{\rho})\times\nabla p + \rho\nabla(\frac{1}{\rho})\times\underline{f} + \nabla\times\underline{f} \\ & + \mu\nabla^2\underline{\Omega} + \rho\nabla(\frac{\lambda+2\mu}{\rho})\times\nabla(\nabla\cdot\underline{v}) - \rho\nabla(\frac{\mu}{\rho})\times\nabla\times\underline{\Omega} \\ & + \rho(\nabla\cdot\underline{v})\nabla(\frac{1}{\rho})\times\nabla\lambda - \nabla\lambda\times\nabla(\nabla\cdot\underline{v}) + 2\nabla\times(\underline{D}\cdot\nabla\mu) \\ & + \rho\nabla(\frac{2}{\rho})\times(\underline{D}\cdot\nabla\mu)\end{aligned} \tag{48b}$$

Taking the integral of Eq. (48a) over a fixed control volume V and applying Green's transformation Eq. (6) and Eqs. (6) and (7) on p. 53 of Ref. 17 we obtain the integral form of the conservation equation for $\rho\underline{\Omega}$ in terms of fixed elements dV and dA

$$\begin{aligned}\frac{\partial}{\partial t}\int_V \rho\underline{\Omega}dV + \oint_A \rho\underline{\Omega}(\underline{v}\cdot\underline{n})dA \\ = \oint_A \rho\underline{v}(\underline{\Omega}\cdot\underline{n})dA + \int_V \underline{Q}_1 dV\end{aligned} \tag{48c}$$

The corresponding conservation equation in terms of moving elements $d\hat{V}$ and $d\hat{A}$ in case $t_1 = t$ is Eq. (16) with

$$\underline{W}=(\rho\underline{\Omega}),\ \underline{\Sigma} = (\rho\underline{v}\ \underline{\Omega}),\ \underline{Q} = (\underline{Q}_1) \quad . \tag{17c}$$

Combining Eqs. (17a), (17b) and (17c) with Eqs. (17) we convert Eq. (16) to a vector equation representing six conservation equations in integral form with

$$\underline{W} = \begin{pmatrix} \rho \\ \rho\underline{v} \\ \rho E \\ \rho H \\ \rho S \\ \rho\underline{\Omega} \end{pmatrix}, \underline{\Sigma} = \begin{pmatrix} 0 \\ \underline{\sigma} \\ \underline{\sigma}\cdot\underline{v} - \underline{q} \\ \underline{\sigma}\cdot\underline{v} - \underline{q} + p\underline{v} \\ \underline{0} \\ \rho\underline{v}\ \underline{\Omega} \end{pmatrix}, \underline{Q} = \begin{pmatrix} 0 \\ \underline{f} \\ \underline{f}\cdot\underline{v} + G \\ \underline{f}\cdot\underline{v} + G + \frac{\partial p}{\partial t} \\ \Phi/T - \nabla\cdot\underline{q}/T + G/T \\ \underline{Q}_1 \end{pmatrix} . \quad (17d)$$

## 3.5 BERNOULLI EQUATION.

As discussed in Ref. 18, Section A,6, aside from the boundary layer, the effects of viscosity and heat conduction can be neglected for the majority of gas dynamics problems. Furthermore, the heat addition, except in problems involving combustion, is either zero or very small. Then under ordinary conditions the gas behaves very much like an ideal fluid (i.e., a nonviscous and nonheat-conducting fluid). Therefore, one of the fundamental problems of gas dynamics is to study the adiabatic flow of an ideal gas.

Let us further assume that the density of the fluid is a function of pressure only (i.e., the fluid is barotropic, Ref. 3, p. 150) and that the external force per unit mass is representable by a potential $\varphi$

$$\underline{f}/\rho = - \nabla\varphi \quad . \qquad (60)$$

Then by using the relation (Ref. 17, p. 44)

$$(\underline{v}\cdot\nabla)\underline{v} = \frac{1}{2}\nabla v^2 - \underline{v} \times (\nabla \times \underline{v}) \qquad (61)$$

and definition of $\underline{\Omega}$ (Eq. (45)) Eq. (40) can be written as

$$\frac{\partial \underline{v}}{\partial t} - \underline{v} \times \underline{\Omega} = \frac{-1}{\rho} \nabla p - \nabla\left(\frac{1}{2} v^2\right) - \nabla \varphi = -\nabla \chi \quad , \tag{62}$$

where

$$\chi = \int \frac{dp}{\rho} + \frac{1}{2} v^2 + \varphi \quad . \tag{63}$$

Let s be a parameter along an arbitrary curve in space. At any point (with position vector $\underline{r}$) along this curve $d\underline{r}/ds$ is a unit tangent vector to the curve. The component of Eq. (62) in the direction of $d\underline{r}/ds$ is

$$\frac{\partial \underline{v}}{\partial t} \cdot \frac{d\underline{r}}{ds} - (\underline{v} \times \underline{\Omega}) \cdot \frac{d\underline{r}}{ds} = - \frac{d\underline{r}}{ds} \cdot \nabla \chi \quad . \tag{64}$$

Since

$$\frac{d\underline{r}}{ds} \cdot \nabla \int \left(\frac{\partial \underline{v}}{\partial t} \cdot \frac{d\underline{r}}{ds}\right) ds = \frac{\partial \underline{v}}{\partial t} \cdot \frac{d\underline{r}}{ds} \tag{65}$$

we may express Eq. (64) as (Ref.19, p. 121)

$$(\underline{v} \times \underline{\Omega}) \cdot \frac{d\underline{r}}{ds} = \frac{d\underline{r}}{ds} \cdot \nabla X \quad , \tag{66}$$

where

$$X = \int \frac{dp}{\rho} + \frac{1}{2} v^2 + \varphi + \int \left(\frac{\partial \underline{v}}{\partial t} \cdot \frac{d\underline{r}}{ds}\right) ds \quad . \tag{67}$$

At any point in time and at every point in the flow field let $d\underline{r}/ds$ represent a tangent vector to a streamline (in the direction of $\underline{v}$) or a vortex line (in the direction of $\underline{\Omega}$).

Then the scalar triple product on the left side of Eq. (66) is zero and the Bernoulli's equation

$$X = \text{a constant} \tag{68}$$

is valid along any streamline or vortex line but may have different constants for different streamlines and vortex lines.

For an irrotational flow $\underline{\Omega} = \underline{0}$ and there is a velocity potential $\phi$:

$$\underline{v} = \nabla\phi \quad , \tag{69}$$

Eq. (68) is valid along every curve in the flow field since in Eq. (66) $\nabla X = \underline{0}$ everywhere. Furthermore, from Eqs. (62) and (69) we obtain the Bernoulli's equation for a potential flow

$$\int \frac{dp}{\rho} + \frac{1}{2} v^2 + \varphi + \frac{\partial \phi}{\partial t} = \text{a constant} \quad , \tag{70}$$

which holds throughout the flow field. The integral in Eq. (70) must be computed with isentropic pressure-density relation because as discussed in the previous section, irrotational motion can generally be maintained only by constant specific entropy throughout the field.

For a perfect gas with constant specific heats, the equation of state can be written in terms of S, p, $\rho$ as [Ref. 18, Eq. (7-16)]

$$p = \text{const}\ \rho^{\gamma} e^{S/c_v} \quad . \tag{71}$$

Then Eq. (70) becomes [Ref. 18, Eq. (8-3)] for $\varphi = 0$ and $S$ = const (isentropic)

$$\frac{\partial\phi}{\partial t} + \frac{1}{2} v^2 + \frac{\gamma}{\gamma-1} \frac{p}{\rho} = \frac{\partial\phi}{\partial t} + H = \text{a constant} . \tag{72}$$

Therefore irrotational flows are not necessarily isoenergetic in the sense that the total energy $H$ is a constant throughout the field. Irrotational flows are isoenergetic only if they are steady, i.e., $\partial\phi/\partial t = 0$.

On the other hand for a rotational flow Eqs. (67) and (68) show that when the motion is steady (i.e., $\partial \underline{v}/\partial t = \underline{0}$) the total energy $H$ is a constant along any streamline or vortex line [compare with the energy equation Eq. (37)]. The difference is that for steady (rotational) isoenergetic flow the motion is not necessarily isentropic, while for steady irrotational (isoenergetic) flow the motion must be isentropic. An example for the isoenergetic but nonisentropic flow is the problem of steady supersonic flow over a body with curved detached shock.

## SECTION 4

## TRANSFORMATION OF CONSERVATION EQUATIONS IN TERMS OF ORTHOGONAL CURVILINEAR COORDINATES

The vectorial forms of the mass, momentum and energy conservation equations Eqs. (32), (33), and (34) can be expressed in a general curvilinear coordinate system by substituting the expressions for the gradient, divergence, and curl operators in such a system given, for example, in Reference 20, Appendix C. For practically useful orthogonal curvilinear coordinates $x_1$, $x_2$, $x_3$, with local unit vectors $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ in the directions of increase of $x_1$, $x_2$, $x_3$ the specific expressions can be found in Ref. 21, Appendix 2.

An elementary displacement can be written as

$$d\underline{s} = h_1 dx_1 \underline{e}_1 + h_2 dx_2 \underline{e}_2 + h_3 dx_3 \underline{e}_3 \quad , \tag{73}$$

where $h_1$, $h_2$ and $h_3$ are the scale factors. The velocity components are $v_1$, $v_2$ and $v_3$ such that

$$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3 \quad . \tag{74}$$

Then Eqs. (32), (33) and (34) can be written as follows (Ref. 16, Section 1.2.2):

<u>mass</u>:

$$\frac{\partial \rho}{\partial t} + \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial x_j} \left( \frac{1}{h_j} h_1 h_2 h_3 \rho v_j \right) = 0 \quad , \tag{75}$$

<u>momentum</u>:

$$\begin{aligned} \frac{\partial}{\partial t} \left( \rho v_1 \right) &+ \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial x_j} \left( \frac{1}{h_j} h_1 h_2 h_3 \mathcal{T}_{1j} \right) + \\ &+ \frac{1}{h_1 h_2} \left( \mathcal{T}_{12} \frac{\partial h_1}{\partial x_2} - \mathcal{T}_{22} \frac{\partial h_2}{\partial x_1} \right) \\ &+ \frac{1}{h_1 h_3} \left( \mathcal{T}_{13} \frac{\partial h_1}{\partial x_3} - \mathcal{T}_{33} \frac{\partial h_3}{\partial x_1} \right) = \underline{f} \cdot \underline{e}_1 \quad , \end{aligned} \tag{76}$$

and two morens by cyclic permütation

**energy:**

$$\frac{\partial}{\partial t} \frac{1}{h_1h_2h_3} \frac{\partial}{\partial x_j} \left\{\frac{1}{h_j} h_1h_2h_3 \left[(\rho E+p) v_j\right.\right.$$

$$\left.\left.\tau_{ij}v_i - k \frac{1}{h_j} \frac{\partial T}{\partial x_j}\right]\right\} = \underline{f} \cdot \underline{v} + G \quad , \tag{77}$$

where the su convention has been used and $\mathcal{T}_{ij}$ are the components o:nsor $\underline{\underline{\mathcal{T}}}$ :

$$\rho \underline{\underline{vv}} - \underline{\underline{\sigma}} = \rho \underline{\underline{vv}} + p\underline{\underline{I}} - \underline{\underline{\tau}} \quad . \tag{78}$$

They can be ed as

$$_{i} = \rho v_i v_j + p\delta_{ij} - \tau_{ij} \tag{79}$$

with

$$_{i} = \lambda (\nabla \cdot \underline{v}) \delta_{ij} + 2\mu \, D_{ij} \quad , \tag{80}$$

where

$$\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{81}$$

is the Kronecibol.

In Eq. (80)

$$\frac{1}{h_1h_2h_3} \frac{\partial}{\partial x_\ell} \left(\frac{1}{h_\ell} h_1h_2h_3v_\ell\right) \quad , \tag{82}$$

$$\frac{1}{h_1}\left(\frac{\partial v_1}{\partial x_1} + \frac{v_2}{h_2} \frac{\partial h_1}{\partial x_2} + \frac{v_3}{h_3} \frac{\partial h_1}{\partial h_3}\right) \quad , \tag{83}$$

with $D_{22}$ and $D_{33}$ following by cyclic permutation and

$$D_{ij(i\neq j)} = \frac{1}{2}\left[\frac{1}{h_i}\frac{\partial v_j}{\partial x_i} + \frac{1}{h_j}\frac{\partial v_i}{\partial x_j} - \frac{1}{h_i h_j}\left(v_i \frac{\partial h_i}{\partial x_j} + v_j \frac{\partial h_j}{\partial x_i}\right)\right] \ . \quad (84)$$

SECTION 5

CONSE EQUATIONS IN CARTESIAN COORDINATES

To the Carcoordinates

$$x_1 = x,\ x_2 = y,\ x_3 = z \qquad (85)$$

there corrthe scale factors

$$h_1 = h_2 = h_3 = 1 \quad . \qquad (86)$$

The conserequations Eqs. (75), (76) and (77) become when Eq. (used:

<u>mass:</u>

$$\frac{\partial}{x_j}(\rho v_j) = 0 \quad , \qquad (87)$$

<u>momentum:</u>

$$_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) + \frac{\partial p}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} = f_i \quad , \qquad (88)$$

<u>energy:</u>

$$) + \frac{\partial}{\partial x_j}\left[(\rho E + p)v_j - \tau_{ij}v_i - k\frac{\partial T}{\partial x_j}\right] = f_i v_i + G \ . \qquad (89)$$

where, sinEqs. (82) to (84)

$$\nabla \cdot \underline{v} = \frac{\partial v_\ell}{\partial x_\ell} \qquad (90)$$

and

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \quad , \qquad (91)$$

we have, f1(80),

$$= \lambda\ \delta_{ij}\frac{\partial v_\ell}{\partial x_\ell} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \quad . \qquad (92)$$

These resue with those usually found in the literature expressed sian tensor notations (see, for example, Ref. 9, Ed9)).

## SECTION 6

## CONSERVATION EQUATIONS IN CYLINDRICAL COORDINATES

To the cylindrical coordinates

$$x_1 = r,\ x_2 = \theta,\ x_3 = z \tag{93}$$

there correspond the scale factors (Reference 22, Appendix 4)

$$h_1 = 1,\ h_2 = r,\ h_3 = 1\ . \tag{94}$$

The conservation equations Eqs. (75), (76) and (77) become

<u>mass</u>:
$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0, \tag{95}$$

<u>momentum</u>:
$$\frac{\partial}{\partial t}(\rho v_r) + \frac{1}{r}\frac{\partial}{\partial r}(r\mathcal{T}_{11}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\mathcal{T}_{12}) + \frac{\partial}{\partial z}(\mathcal{T}_{13})$$

$$- \frac{1}{r}\mathcal{T}_{22} = f_r\ , \tag{96}$$

$$\frac{\partial}{\partial t}(\rho v_\theta) + \frac{1}{r}\frac{\partial}{\partial r}(r\mathcal{T}_{21}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\mathcal{T}_{22}) + \frac{\partial}{\partial z}(\mathcal{T}_{23})$$

$$+ \frac{1}{r}\mathcal{T}_{21} = f_\theta\ , \tag{97}$$

$$\frac{\partial}{\partial t}(\rho v_z) + \frac{1}{r}\frac{\partial}{\partial r}(r\mathcal{T}_{31}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\mathcal{T}_{32})$$

$$+ \frac{\partial}{\partial z}(\mathcal{T}_{33}) = f_z\ , \tag{98}$$

<u>energy</u>:
$$\frac{\partial}{\partial t}(\rho E) + \frac{1}{r}\frac{\partial}{\partial r}\left\{ r\left[ (\rho E + p)v_r - (\tau_{11}v_r + \tau_{21}v_\theta + \tau_{31}v_z) \right.\right.$$

$$\left.\left. - k\frac{\partial T}{\partial r}\right]\right\}$$

$$+ \frac{1}{r}\frac{\partial}{\partial\theta}\Big[(\rho E+p)v_\theta - (\tau_{12}v_r+\tau_{22}v_\theta+\tau_{32}v_z)$$

$$- \frac{k}{r}\frac{\partial T}{\partial\theta}\Big]$$

$$+ \frac{\partial}{\partial z}\Big[(\rho E+p)v_z - (\tau_{13}v_r+\tau_{23}v_\theta+\tau_{33}v_z)$$

$$- k\frac{\partial T}{\partial z}\Big]$$

$$= f_r v_r + f_\theta v_\theta + f_z v_z + G \quad , \tag{99}$$

where $\mathcal{T}_{ij}$ is given by Eq. (79) with $\tau_{ij}$ given by Eq. (80) in which, according to Eqs. (82) to (84),

$$\nabla\cdot\underline{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z} \quad , \tag{100}$$

and

$$\left.\begin{aligned}
D_{11} &= \frac{\partial v_r}{\partial r} \, , \; D_{22} = \frac{1}{r}\left(\frac{\partial v_\theta}{\partial\theta} + v_r\right) \, , \; D_{33} = \frac{\partial v_z}{\partial z} \, , \\
D_{12} &= D_{21} = \frac{1}{2}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right) \, , \\
D_{23} &= D_{32} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta} + \frac{\partial v_\theta}{\partial z}\right) \, , \\
D_{31} &= D_{13} = \frac{1}{2}\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) \, .
\end{aligned}\right\} \tag{101}$$

## SECTION 7

## CONSERVATION EQUATIONS IN SPHERICAL COORDINATES

To the spherical coordinates

$$x_1 = r, \quad x_2 = \theta, \quad x_3 = \varphi \tag{102}$$

there correspond the scale factors (Reference 21, Appendix 2)

$$h_1 = 1, \quad h_2 = r\,, \quad h_3 = r \sin\theta\ . \tag{103}$$

The conservation equations Eqs. (75), (76) and (77) become

<u>mass</u>:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta \rho v_r) + \frac{\partial}{\partial \theta}(r \sin\theta \rho v_\theta) + \frac{\partial}{\partial \varphi}(r \rho v_\varphi)\right] = 0\ , \tag{104}$$

<u>momentum</u>:

$$\frac{\partial}{\partial t}(\rho v_r) + \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta\, \mathcal{T}_{11}) + \frac{\partial}{\partial \theta}(r \sin\theta\, \mathcal{T}_{12}) + \frac{\partial}{\partial \varphi}(r \mathcal{T}_{13})\right] - \frac{1}{r}\mathcal{T}_{22} - \frac{1}{r}\mathcal{T}_{33} = f_r\ , \tag{105}$$

$$\frac{\partial}{\partial t}(\rho v_\theta) + \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta\, \mathcal{T}_{21}) + \frac{\partial}{\partial \theta}(r \sin\theta\, \mathcal{T}_{22}) + \frac{\partial}{\partial \varphi}(r \mathcal{T}_{23})\right] - \frac{\cos\theta}{r \sin\theta}\mathcal{T}_{33} - \frac{1}{r}\mathcal{T}_{21} = f_\theta\ , \tag{106}$$

$$\frac{\partial}{\partial t}(\rho v_\theta) + \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta\, \mathcal{T}_{31}) + \frac{\partial}{\partial \theta}(r \sin\theta\, \mathcal{T}_{32}) + \frac{\partial}{\partial \varphi}(r \mathcal{T}_{33})\right] + \frac{1}{r}\mathcal{T}_{31} + \frac{\cos\theta}{r \sin\theta}\mathcal{T}_{32} = f_\varphi\ , \tag{107}$$

$$\underline{\text{energy}}: \quad \frac{\partial}{\partial t}(\rho E) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left\{r^2\sin\theta\left[(\rho E+p)v_r - (\tau_{11}v_r+\tau_{21}v_\theta+\tau_{31}v_\varphi)\right.\right.$$

$$\left.\left. - k\frac{\partial T}{\partial r}\right]\right\} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left\{r\sin\theta\left[(\rho E+p)v_\theta - (\tau_{12}v_r+\tau_{22}v_\theta+\tau_{32}v_\varphi)\right.\right.$$

$$\left.\left. - \frac{k}{r}\frac{\partial T}{\partial\theta}\right]\right\} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\varphi}\left\{r\left[(\rho E+p)\ v_\varphi - (\tau_{13}v_r+\tau_{23}v_\theta+\tau_{33}v_\varphi)\right.\right.$$

$$\left.\left. - \frac{k}{r\sin\theta}\frac{\partial T}{\partial\varphi}\right]\right\} = f_r v_r + f_\theta v_\theta + f_\varphi v_\varphi + G\ , \qquad (108)$$

where $\mathcal{T}_{ij}$ is given by Eq. (79) with $\tau_{ij}$ given by Eq. (80), in which, according to Eqs. (82) to (84),

$$\nabla\cdot\underline{v} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta v_r) + \frac{\partial}{\partial\theta}(r\sin\theta v_\theta) + \frac{\partial}{\partial\varphi}(rv_\varphi)\right]\ , \qquad (109)$$

and

$$\left.\begin{aligned}
&D_{11} = \frac{\partial v_r}{\partial r}\ ,\ D_{22} = \frac{1}{r}\left(\frac{\partial v_\theta}{\partial\theta} + v_r\right),\ D_{33} = \frac{1}{r\sin\theta}\left(\frac{\partial v_\varphi}{\partial\varphi} + \sin\theta v_r + \cos\theta v_\theta\right), \\
&D_{12} = D_{21} = \frac{1}{2}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\ , \\
&D_{23} = D_{32} = \frac{1}{2r\sin\theta}\left(\sin\theta\frac{\partial v_\varphi}{\partial\theta} + \frac{\partial v_\theta}{\partial\varphi} - \cos\theta v_\varphi\right), \\
&D_{31} = D_{13} = \frac{1}{2r\sin\theta}\left(\frac{\partial v_r}{\partial\varphi} + r\sin\theta\frac{\partial v_\varphi}{\partial r} - \sin\theta v_\varphi\right).
\end{aligned}\right\}\ (110)$$

## SECTION 8

## COMPACT ONE-DIMENSIONAL FORMULATION

The conservation equations in Sections 5, 6 and 7 for three-dimensional Cartesian, cylindrical and spherical problems respectively, can be readily reduced to those for various one- and two-dimensional problems by setting appropriate velocity components, force components and derivatives to zero. In particular, if we consider the one-dimensional problems in which

$$x_1 = r,\ \underline{v} = \underline{e}_1 v_r(r),\ v_2 = v_3 = 0,\ \frac{\partial(\ )}{\partial x_2} = \frac{\partial(\ )}{\partial x_3} = 0,\ f_2 = f_3 = 0\ , \tag{111}$$

the conservation equations in Cartesian, cylindrical and spherical coordinates can be compactly expressed in one single set of equations (with $j = 0, 1, 2$ representing plane, line and point symmetry respectively):

<u>mass:</u>

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^j}\frac{\partial}{\partial r}\left(r^j \rho v_r\right) = 0\ , \tag{112}$$

<u>momentum:</u>

$$\frac{\partial}{\partial t}(\rho v_r) + \frac{1}{r^j}\frac{\partial}{\partial r}\left(r^j \rho v_r{}^2\right)$$

$$= f_r - \frac{\partial p}{\partial r} + \frac{\partial}{\partial r}\left[\frac{(\lambda+2\mu)}{r^j}\ \frac{\partial(r^j v_r)}{\partial r}\right] - \frac{2j}{r} v_r \frac{\partial \mu}{\partial r}\ , \tag{113}$$

<u>energy:</u>

$$\frac{\partial}{\partial t}(\rho E) + \frac{1}{r^j}\frac{\partial}{\partial r}\left[r^j v_r(\rho E + p)\right]$$

$$= f_r v_r + \frac{1}{r^j}\frac{\partial}{\partial r}\left[r^j v_r\left\{\frac{\lambda}{r^j}\frac{\partial}{\partial r}\left(r^j v_r\right) + 2\mu\frac{\partial v_r}{\partial r}\right\}\right]$$

$$+ \frac{1}{r^j}\frac{\partial}{\partial r}\left(k r^j \frac{\partial T}{\partial r}\right) + G\ . \tag{114}$$

## SECTION 9

## CONCLUSIONS

Following a survey of the basic equations, the mass, momentum and energy conservation equations and the components of the deformation and stress tensors for a nonsteady flow of a compressible viscous single-phase fluid have been expressed in vector and dyadic notations, transformed in terms of orthogonal curvilinear coordinates and specialized for the three-dimensional cases in Cartesian, cylindrical and spherical coordinates. These equations can then be reduced to corresponding equations for one- and two-dimensional flows in various coordinates. They are in a format readily applicable to practical problems.

If desired the procedure can be extended to general nonorthogonal curvilinear coordinate systems.

This report also contains a generalization of the conservation equations in integral form based on an extension of the Reynolds transport theorem.

SECTION 10

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